

# SOBOLEV INDEX: A CLASSIFICATION OF LÉVY PROCESSES VIA THEIR SYMBOLS

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**ABSTRACT.** We classify Lévy processes according to the solution spaces of the associated parabolic PIDEs. This classification reveals structural characteristics of the processes and is relevant for applications such as for solving PIDEs numerically for pricing options in Lévy models.

The classification is done via the Fourier transform i.e. via the symbol of the process. We define the Sobolev index of a Lévy process by a certain growth condition on the symbol. It follows that for Lévy processes with Sobolev index  $\alpha$  the corresponding evolution problem has a unique weak solution in the Sobolev-Slobodeckii space  $H^{\alpha/2}$ . We show that this classification applies to a wide range of processes. Examples are the Brownian motion with or without drift, generalised hyperbolic (GH), CGMY and (semi) stable Lévy processes.

A comparison of the Sobolev index with the Blumenthal-Gettoor index sheds light on the structural implication of the classification. More precisely, we discuss the Sobolev index as an indicator of the smoothness of the distribution and of the variation of the paths of the process. This highlights the relation between the  $p$ -variation of the paths and the degree of smoothing effect that stems from the distribution.

## 1. INTRODUCTION

The Feynman-Kac formula provides a fundamental link between conditional expectations and solutions to PDEs. Under suitable regularity assumptions, a Feynman-Kac representation relates certain conditional expectations to weak solutions of parabolic equations. In financial mathematics this fact is used to compute option prices by solving parabolic equations.

In the context of Lévy processes, conditional expectations are linked to solutions of Partial Integro Differential Equations (PIDEs). In (Matache, von Petersdorff, and Schwab 2004), (Matache, Schwab, and Wihler 2005), (Matache, Nitsche, and Schwab 2005) wavelet-Galerkin methods for pricing European and American options have been developed. The methods have been extended to multivariate models, see (Reich, Schwab, and Winter 2010), (Winter 2009) and the references therein. Also standard finite element methods are efficiently used for pricing basket options, even in high dimensional models using dimension reduction techniques, see (Heppenger 2010) and (Heppenger 2012). (Achdou 2008) provides a calibration procedure of a Lévy model based on PIDEs. Essential for those finite element methods is the existence

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and uniqueness of a weak solution of the PIDE, related to the underlying process, in a certain Sobolev-Slobodeckii space.

In other words, a relation between Lévy processes and Sobolev-Slobodeckii spaces is seminal. More precisely, Lévy processes of a certain type are linked to Sobolev-Slobodeckii spaces  $H^s$  with a certain index  $s > 0$ . This index is important since it classifies the nature of the related evolution problems, resp. of its weak solutions. It turns out that if the symbol of the Lévy process satisfies certain polynomial growth conditions with degree  $2s$ , then the evolution problem has a weak solution in the space  $H^s$ . The structural connection between certain types of Lévy processes and Sobolev-Slobodeckii spaces is thus reflected by the index  $s$ . This leads us to the definition the *Sobolev index* of the Lévy process.

It is worth mentioning that in the classical theory on weak solutions of evolution problems, existence and uniqueness of a weak solution are related to the so-called Gårding and continuity inequalities of the bilinear form. The bilinear form is given via the operator of the equation. While PIDEs are classified via their operators, Lévy processes are determined by their characteristic functions due to the famous Lévy-Khintchine formula. Various classes of Lévy processes, as e.g. the CGMY processes, are directly defined by specifying their characteristic functions. The symbol of a Lévy process is given via the exponent of the Fourier transform of the process, resp. in terms of the cumulant generating function, see e.g. (Jacob 2001). Therefore properties of the symbol can be canonically derived for a wide range of Lévy processes.

Crucial for connecting both approaches is Parseval's equality that allows to express the bilinear form associated to the infinitesimal generator via the symbol; details are provided in Section 2, where the notation and this connection is formally shown. In Section 3 the argument is outlined in detail.

For various classes of Lévy processes we compute the Sobolev index in Section 4. The Brownian motion with and without drift has Sobolev index 2. We show that the generalised hyperbolic (GH) processes, Cauchy processes, Student- $t$  processes, and the multivariate NIG processes have Sobolev index 1. The Sobolev index is additionally discussed for CGMY processes, and for Lévy processes without continuous martingale part which have an absolutely continuous Lévy measure.

The symbol of a generic  $\alpha$ -stable Lévy process is of the form  $c|u|^\alpha$  with a positive constant  $c$ , hence it is polynomial and the Sobolev index can be deduced in an obvious way. In Section 5, we will shed light on the Sobolev index for the wider class of  $\alpha$ -semi-stable Lévy processes.

The last section is dedicated to the examination of the Sobolev index in connection to the Blumenthal-Gettoor index. For Lévy processes that have a Sobolev index smaller than 2, we derive that the Blumenthal-Gettoor index is bigger or equal to the Sobolev index. Thereby a link is established between path properties of the process and the smoothing effect of the related evolution problem. Moreover, in view of the Feynman-Kac representation a link to the smoothing effect of the distribution is provided.

## 2. THE INFINITESIMAL GENERATOR AND THE SYMBOL OF A LÉVY PROCESS

Let  $L$  be a Lévy process with values in  $\mathbb{R}^d$  and characteristics  $(b, \sigma, F)$  with respect to a truncation function  $h$ . Here, a measurable function  $h : \mathbb{R}^d \rightarrow \mathbb{R}$  is called a truncation function if  $h(x) = x$  in a neighbourhood of 0.

The distribution of the process is uniquely determined by the distribution  $\mu_t := P^{L_t}$  for any (for some)  $t > 0$  and hence by the characteristic function  $\hat{\mu}_t$  of  $L_t$ ,

$$\hat{\mu}_t(\xi) = E e^{i\langle \xi, L_t \rangle} = e^{t\theta(i\xi)} . \quad (1)$$

with cumulant generating function

$$\theta(i\xi) = -\frac{1}{2}\langle \xi, \sigma \xi \rangle + i\langle \xi, b \rangle + \int \left( e^{i\langle \xi, y \rangle} - 1 - i\langle \xi, h(y) \rangle \right) F(dy) , \quad (2)$$

where we denote by  $\langle \cdot, \cdot \rangle$  the Euclidean scalar product in  $\mathbb{R}^d$ . The matrix  $\sigma$  is a symmetric, positive semidefinite  $d \times d$ -matrix,  $b \in \mathbb{R}^d$  and  $F$  is a Lévy measure i.e. a Borel measure on  $\mathbb{R}^d$  with  $\int (|x|^2 \wedge 1) F(dx) < \infty$ .

Furthermore we denote by  $\mathcal{G}$  the infinitesimal generator of the process  $L$ , i.e.

$$\begin{aligned} \mathcal{G} f(x) &= \frac{1}{2} \sum_{j,k=1}^d \sigma^{j,k} \frac{\partial^2 f}{\partial x_j \partial x_k}(x) + \sum_{j=1}^d b^j \frac{\partial f}{\partial x_j}(x) \\ &\quad + \int_{\mathbb{R}^d} \left( f(x+y) - f(x) - \sum_{j=1}^d \frac{\partial f}{\partial x_j}(x) (h(y))_j \right) F(dy) \end{aligned} \quad (3)$$

for  $f \in C_0^2(\mathbb{R}^d)$ . We define

$$\mathcal{A} := -\mathcal{G} .$$

The *symbol*  $A$  of the process  $L$  is defined by

$$\begin{aligned} A(\xi) &:= \frac{1}{2}\langle \xi, \sigma \xi \rangle + i\langle \xi, b \rangle - \int \left( e^{-i\langle \xi, y \rangle} - 1 + i\langle \xi, h(y) \rangle \right) F(dy) \\ &= -\theta(-i\xi) , \end{aligned}$$

compare e.g. (Jacob 2001). We have

$$\hat{\mu}_t(\xi) = E e^{i\langle \xi, L_t \rangle} = e^{-tA(-\xi)} . \quad (4)$$

Let us further denote by  $S(\mathbb{R}^d)$  the Schwartz space i.e. the set of smooth functions  $\varphi \in C^\infty(\mathbb{R}^d, \mathbb{C})$  with

$$(1 + |x|^m) |D^\alpha \varphi(x)| \rightarrow 0, |x| \rightarrow \infty$$

for every multi index  $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}_0^d$  and every  $m \in \mathbb{N}_0$ , where  $D^\alpha$  denotes the multiple partial derivative

$$D^\alpha \varphi(x) := \frac{\partial^{\alpha_1} \dots \partial^{\alpha_d}}{\partial x_1^{\alpha_1} \dots \partial x_d^{\alpha_d}} \varphi(x) .$$

Let us sketch a relation between the Fourier transform of the distribution, the symbol of the process and a partial integro differential equation. Let  $u \in S(\mathbb{R}^d)$  and

$$T_t u(x) := E_x(u(L_t)) := E(u(L_t + x))$$

If the absolute value of the characteristic function  $\hat{\mu} : \mathbb{R}^d \rightarrow \mathbb{C}$  is bounded by a polynomial, then Parseval's equality yields

$$T_t u(x) = \frac{1}{(2\pi)^d} \int e^{-i\langle \xi, x \rangle} \hat{\mu}_t(-\xi) \hat{u}(\xi) d\xi .$$

In particular,  $(T_t)_{t \geq 0}$  is a family of pseudo differential operators  $T_t$  with symbol  $\hat{\mu}_t(\cdot)$ . Changing the order of integration and differentiation, we obtain

$$\mathcal{G}u(x) = \lim_{t \rightarrow 0} \frac{T_t u - u}{t}(x) = \partial_t (T_t u(x))|_{t=0} = \frac{1}{(2\pi)^d} \int e^{-i\langle x, \xi \rangle} A(\xi) \hat{u}(\xi) d\xi,$$

where  $\hat{u}$  denotes the Fourier transform of  $u$ . Hence the infinitesimal generator  $\mathcal{G}$ , which satisfies

$$\mathcal{G}u = \lim_{t \rightarrow 0} \frac{T_t u - u}{t},$$

compare e.g. Dynkin (1965) and Jacob (2001, Chapter 4) is a pseudo differential operator with symbol  $-A$  resp.

$$\mathcal{A}u(x) = -\mathcal{G}u(x) = \frac{1}{(2\pi)^d} \int e^{-i\langle x, \xi \rangle} A(\xi) \hat{u}(\xi) d\xi.$$

Let us first notice that the symbol  $A$  of a Lévy process is a Borel measurable function  $A : \mathbb{R}^d \rightarrow \mathbb{C}$  and there exists a positive constant  $C > 0$  such that

$$|A(\xi)| \leq C(1 + |\xi|)^2 \quad (\text{for all } \xi \in \mathbb{R}^d), \quad (5)$$

which is well-known and standard to verify. According to the notation in (Eskin 1981), we say that  $A \in S_2^0$ . More generally, we write  $A \in S_\alpha^0$ , if  $|A(\xi)| \leq C(1 + |\xi|)^\alpha$  for a certain  $\alpha \in \mathbb{R}$  and a constant  $C \geq 0$ .

Let  $g \in S(\mathbb{R}^d)$  and

$$v(t, x) := E(g(L_T) \mid L_t = x),$$

then

$$v(t, x) = E(g(L_{T-t} + x)) = \frac{1}{(2\pi)^d} \int e^{-i\langle \xi, x \rangle} \hat{\mu}_{T-t}(-\xi) \hat{g}(\xi) d\xi \quad (6)$$

and hence  $\hat{v}(t, \xi) = e^{(T-t)A(\xi)} \hat{g}(\xi)$ . On the other hand we have

$$\begin{aligned} \partial_t v(t, x) &= \partial_t (T_{T-t} g(x)) = \frac{1}{(2\pi)^d} \int e^{-i\langle x, \xi \rangle} \left( \partial_s e^{(T-s)A(\xi)} \Big|_{s=t} \right) \hat{g}(\xi) d\xi \\ &= \frac{1}{(2\pi)^d} \int e^{-i\langle x, \xi \rangle} A(\xi) \hat{v}(t, \xi) d\xi \\ &= -\mathcal{G}v(t, x). \end{aligned}$$

In other words, the function  $v$  satisfies the PIDE

$$\begin{aligned} \partial_t v(t, x) + \mathcal{G}v(t, x) &= 0 \quad \text{for all } (t, x) \in (0, T) \times \mathbb{R}^d \\ v(T, x) &= g(x) \quad \text{for all } x \in \mathbb{R}^d. \end{aligned}$$

For  $V(t, x) := v(T - t, x) = E(g(L_T) \mid L_{T-t} = x)$  we accordingly have

$$\partial_t V(t, x) + \mathcal{A}V(t, x) = 0 \quad \text{for all } (t, x) \in (0, T) \times \mathbb{R}^d \quad (7)$$

$$V(0, x) = g(x) \quad \text{for all } x \in \mathbb{R}^d. \quad (8)$$

In this case the function  $v$  solves the PIDE in the classical sense i.e. point wise. Beyond that, in cases where a point wise solution may fail to exist, a Feynman-Kac formula ties together weak solutions of certain PIDEs and conditional expectations, see (Bensoussan and Lions 1982).

## 3. DEFINITION OF THE SOBOLEV INDEX

According to inequality (5), the symbol  $A$  belongs to  $S_2^0$ , we have  $A\hat{u} \in L^2(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$  for every function  $u \in S(\mathbb{R}^d)$  and the Fourier inverse of  $A\hat{u}$  is well defined. Moreover an elementary calculation shows that

$$\frac{1}{(2\pi)^d} \int A(\xi) \hat{u}(\xi) e^{-i\langle x, \xi \rangle} d\xi = \mathcal{A}u(x) \quad \text{for all } x \in \mathbb{R}^d \text{ and all } u \in S(\mathbb{R}^d). \quad (9)$$

Equation (9) coincides with the definition of a *pseudo differential operator*  $\mathcal{A}$  with symbol  $A \in S_2^0$ . In other words, we have checked that  $A$  is indeed the symbol of the so called pseudo differential operator (PDO)  $\mathcal{A}$ .

**Remark 3.1.** Let  $L$  be a Lévy process with infinitesimal generator  $\mathcal{G}$ . Since the PDO  $\mathcal{A} = -\mathcal{G}$  is real-valued the associated symbol  $A$  satisfies

$$A(\xi) = \overline{A(-\xi)} \quad \text{for all } \xi \in \mathbb{R}^d.$$

In the sequel we will work with *Sobolev-Slobodeckii spaces*. These are defined by

$$H^s(\mathbb{R}^d) = \{u \in S'(\mathbb{R}^d) \mid \hat{u} \in L_{\text{loc}}^1(\mathbb{R}^d, \mathbb{C}^d) \text{ with } \|u\|_s^2 < \infty\}$$

for  $s \in \mathbb{R}$  with

$$\|u\|_s^2 = \int |\hat{u}(\xi)|^2 (1 + |\xi|)^{2s} d\xi,$$

where  $S'(\mathbb{R}^d)$  denotes the space of *generalised functions* i.e. the dual space of the Schwartz space  $S(\mathbb{R}^d)$ .

The following assertion is taken from (Eskin 1981, Lemma 4.4). To keep our presentation self contained we include the short but crucial proof.

**Lemma 3.2.** Let  $A \in S_\alpha^0$  with PDO  $\mathcal{A}$ . Then there exists a constant  $C \geq 0$ , such that

$$\|\mathcal{A}u\|_{s-\alpha} \leq C\|u\|_s \quad \text{for all } u \in S(\mathbb{R}^d)$$

for every  $s \in \mathbb{R}$ . Furthermore the operator  $\mathcal{A} : S(\mathbb{R}^d) \rightarrow C^\infty(\mathbb{R}^d, \mathbb{C})$  has a unique linear and continuous extension

$$\mathcal{A} : H^s(\mathbb{R}^d) \rightarrow H^{s-\alpha}(\mathbb{R}^d).$$

*Proof.* From the definition of the norm and since  $A \in S_\alpha^0$ , we conclude

$$\begin{aligned} \|\mathcal{A}u\|_{s-\alpha}^2 &= \int (1 + |\xi|)^{2(s-\alpha)} |A(\xi) \hat{u}(\xi)|^2 d\xi \\ &\leq C \int (1 + |\xi|)^{2s} |\hat{u}(\xi)|^2 d\xi \\ &= C\|u\|_s^2. \end{aligned}$$

Obviously  $\mathcal{A}u \in C^\infty$  holds for every  $u \in S(\mathbb{R}^d)$  and since  $S(\mathbb{R}^d)$  is dense in  $H^s(\mathbb{R}^d)$  there exists a unique linear and continuous extension  $\mathcal{A} : H^s(\mathbb{R}^d) \rightarrow H^{s-\alpha}(\mathbb{R}^d)$ .  $\square$

For each  $s \in \mathbb{R}$  the dual space  $(H^s(\mathbb{R}^d))^*$  of the Sobolev-Slobodeckii space  $H^s(\mathbb{R}^d)$  is isomorphic to  $H^{-s}(\mathbb{R}^d)$ , compare (Eskin 1981, S. 62, 63). Together with Lemma 3.2 this leads to

**Proposition 3.3.** *If  $\mathcal{A}$  is a PDO with symbol  $A \in S_\alpha^0$ , then*

$$\mathcal{A} : H^s(\mathbb{R}^d) \longrightarrow (H^s(\mathbb{R}^d))^*$$

*is continuous for  $s = \alpha/2$  and the associated bilinear form  $a : H^s(\mathbb{R}^d) \times H^s(\mathbb{R}^d) \rightarrow \mathbb{C}$  defined by*

$$a(u, v) := (\mathcal{A}u)(v)$$

*is continuous on  $H^s(\mathbb{R}^d)$  i.e. there exists a constant  $c > 0$  with*

$$|a(u, v)| \leq c \|u\|_s \|v\|_s \quad \text{for all } u, v \in H^s(\mathbb{R}^d).$$

Let us now observe that for a PDO  $\mathcal{A}$  with symbol  $A \in S_\alpha^0$  and bilinear form  $a$  we have

$$a(u, v) = \int (\mathcal{A}u)(x) \overline{v(x)} dx = \int A(\xi) \hat{u}(\xi) \overline{\hat{v}(\xi)} d\xi \quad (10)$$

for every  $u, v \in S(\mathbb{R}^d)$  by Parseval's identity. Since the bilinear form is expressed in terms of the symbol, the coercivity and the Gårding inequality translate to properties of the symbol. We will study coercivity and Gårding inequality with respect to Sobolev-Slobodeckii spaces.

Let  $A \in S_\alpha^0$  and assume the existence of a positive constant  $c_1$  with

$$\Re(A(\xi)) \geq c_1(1 + |\xi|)^\alpha \quad \text{for all } \xi \in \mathbb{R}^d. \quad (11)$$

Then for any  $u \in S(\mathbb{R}^d)$

$$\Re(a(u, u)) = \int \Re(A(\xi)) |\hat{u}(\xi)|^2 d\xi \geq c_1 \int (1 + |\xi|)^\alpha |\hat{u}(\xi)|^2 d\xi = c_1 \|u\|_{\alpha/2}^2.$$

With the density of  $S(\mathbb{R}^d)$  in  $H^{\alpha/2}(\mathbb{R}^d)$ , the coercivity of the bilinear form  $a$  with respect to the Hilbert space  $H^{\alpha/2}(\mathbb{R}^d)$  follows. Hence, if the symbol  $A \in S_\alpha^0$  of a Lévy process  $L$  satisfies the coercivity condition (11), the infinitesimal operator of  $L$  is elliptic. Moreover the corresponding parabolic equation has a unique solution in the Sobolev-Slobodeckii space  $H^{\alpha/2}(\mathbb{R}^d)$ , which will be discussed in detail in Theorem 3.8.

Let us point out that in contrast to the usual assumptions on a symbol, compare e.g. estimate (B.2) in (Jacob 2005), we do not require any order of differentiability of the symbol. It is well known that the natural domain of the pseudo differential operator  $\mathcal{A}$  is the  $\psi$ -Bessel potential space

$$H_p^{\psi,2}(\mathbb{R}^d) = \left\{ u \in L^2(\mathbb{R}^d) \left| \int_{\mathbb{R}^d} (1 + \psi(\xi))^2 |\hat{u}(\xi)|^2 d\xi < \infty \right. \right\}$$

for  $\psi(\xi) := \theta(i\xi) = -A(-\xi)$ , that is studied in detail in (Farkas, Jacob, and Schilling 2001). We are equally interested in the ellipticity of the operator, hence we investigate also the Gårding inequality.

Notice that the real part of the symbol of a Lévy process is nonnegative,

$$\Re(A(\xi)) = \langle \xi, \sigma \xi \rangle - \int (\cos(\langle x, \xi \rangle) - 1) F(dy) \geq 0. \quad (12)$$

It is straightforward to verify that the space  $H^{\Re(A)} := \overline{C_0^\infty(\mathbb{R}^d)^{\|\cdot\|_{\Re(A)}}}$ , that is the completion of  $C_0^\infty(\mathbb{R}^d, \mathbb{R})$  with respect to the norm  $\|\cdot\|_{\Re(A)}$  given by

$$\|u\|_{\Re(A)} := \int_{\mathbb{R}^d} (1 + \Re(A(\xi))) |\hat{u}(\xi)|^2 d\xi,$$

is a Hilbert space. Moreover,  $H^{\Re(A)} \hookrightarrow L^2(\mathbb{R}^d) \hookrightarrow (H^{\Re(A)})^*$  is a Gelfand triplet, where  $(H^{\Re(A)})^*$  denotes the dual space of  $H^{\Re(A)}$ . For  $u \in C_0^\infty(\mathbb{R}^d, \mathbb{R})$  it follows

$$a(u, u) = \int_{\mathbb{R}^d} A(\xi) |\hat{u}(\xi)|^2 d\xi = \int_{\mathbb{R}^d} \Re(A(\xi)) |\hat{u}(\xi)|^2 d\xi = \|u\|_{\Re(A)}^2 - \|u\|_{L^2}^2.$$

If we assume

$$|\Im(A(\xi))| \leq c(1 + \Re(A(\xi))) \quad \text{for all } \xi \in \mathbb{R}^d \quad (13)$$

with some positive constant  $c \geq 0$ , then we obtain for  $u, v \in C_0^\infty(\mathbb{R}^d, \mathbb{R})$

$$\begin{aligned} |a(u, v)| &= \left| \int_{\mathbb{R}^d} A(\xi) \hat{u}(\xi) \overline{\hat{v}(\xi)} d\xi \right| \\ &\leq \left| \int_{\mathbb{R}^d} \Re(A(\xi)) \hat{u}(\xi) \overline{\hat{v}(\xi)} d\xi \right| + \left| \int_{\mathbb{R}^d} \Im(A(\xi)) \hat{u}(\xi) \overline{\hat{v}(\xi)} d\xi \right| \\ &\leq (1 + c) \int_{\mathbb{R}^d} |(1 + \Re(A(\xi)))| |\hat{u}(\xi) \overline{\hat{v}(\xi)}| d\xi \\ &\leq (1 + c) \|u\|_{\Re(A)} \|v\|_{\Re(A)}. \end{aligned}$$

From the classical result on existence and uniqueness of solutions of parabolic differential equations, compare e.g. (Wloka 1987), we obtain the following result.

**Theorem 3.4.** *Let  $A$  be the symbol of the Lévy process  $L$ . Assume (13). Then, the bilinear form  $a$  is continuous w.r.t.  $H^{\Re(A)}$  and satisfies a Gårding inequality w.r.t.  $H^{\Re(A)}, L^2(\mathbb{R}^d)$ . In particular, the PIDE*

$$\begin{aligned} \dot{u} + \mathcal{A}u &= f \\ u(0) &= g \end{aligned}$$

with  $f \in L^2(0, T; (H^{\Re(A)}(\mathbb{R}^d))^*)$  and initial condition  $g \in L^2(\mathbb{R}^d)$  has a unique solution  $u \in W^1(0, T; H^{\Re(A)}, L^2(\mathbb{R}^d))$ .

For a given Gelfand triplet  $V \hookrightarrow H \hookrightarrow V^*$ , the space  $W^1(0, T; V, H)$  consists of those functions  $u \in L^2(0, T; V)$  that have a derivative  $\partial_t u$  with respect to time in a distributional sense that belongs to the space  $L^2(0, T; V^*)$ . For a Hilbert space  $H$ , the space  $L^2(0, T; H)$  denotes the space of functions  $u : [0, T] \rightarrow H$ , that are weakly measurable and that satisfy  $\int_0^T \|u(t)\|_H^2 dt < \infty$ . For the definition of weak measurability and for a detailed introduction of the space  $W^1(0, T; V, H)$  that relies on the Bochner integral, we refer to the book of (Wloka 1987).

In the following, we focus on the case that the space  $H^{\Re(A)}$  is a Sobolev-Slobodeckii space  $H^s(\mathbb{R}^d)$ , i.e. to the case that the function  $\Re(A)$  in the definition of the space  $H^{\Re(A)}$  can be replaced by a polynomial  $|\xi|^\alpha$  with  $\alpha \in (0, 1]$ . One major advantage of these more concrete spaces is that the index of a Sobolev-Slobodeckii space indicates a certain degree of smoothness. This leads us to define the Sobolev index of a PDO resp. of a Lévy process.

**Definition 3.5.** Let  $A$  be a PDO with symbol  $A$ . We say  $\alpha \in (0, 2]$  is the Sobolev index of the symbol  $A$ , if for all  $\xi \in \mathbb{R}^d$

$$\begin{aligned} |A(\xi)| &\leq C_1 (1 + |\xi|^2)^{\alpha/2} && (\text{Continuity condition}) \text{ and} \\ \Re(A(\xi)) &\geq C_2 |\xi|^\alpha - C_3 (1 + |\xi|^2)^{\beta/2} && (\text{Gårding condition}) \end{aligned}$$

for some  $0 \leq \beta < \alpha$  and constants  $C_1, C_3 \geq 0$  and  $C_2 > 0$ .

If  $L$  is a Lévy process with symbol  $A$  and Sobolev index  $\alpha$ , we call  $\alpha$  the Sobolev index of the Lévy process  $L$ .

Let us notice that the Gårding condition is an assumption on the asymptotic behaviour of the real part of the symbol for large values of  $\xi$ . In case of continuity of  $\xi \rightarrow A(\xi)$ , it is equivalent to the existence of a number  $N > 0$ , such that

$$\Re(A(\xi)) \geq C_2 |\xi|^\alpha \quad \text{for all } |\xi| > N.$$

Not every Lévy process has a Sobolev index, compare Example 4.8. But for important classes of Lévy processes we will show its existence in Section 4 and 5.

**Proposition 3.6.** If the Lévy process has Sobolev index  $\alpha > 0$ , then for every  $t > 0$ , the measure  $\mu_t = P^{L_t}$  has a smooth and bounded density w.r.t. the Lebesgue measure.

*Proof.* The Fourier transform of the measure  $\mu_t$  is given by  $\hat{\mu}_t(\xi) = e^{-tA(-\xi)}$  and

$$|\hat{\mu}_t(\xi)| = e^{-t\Re(A(-\xi))} \leq e^{-C_2 t |\xi|^\alpha + C_3 t (1 + |\xi|^2)^{\beta/2}}$$

with  $C_2 > 0$ ,  $C_3 \geq 0$  and  $0 \leq \beta < \alpha$  by assumption. This shows that the term  $|\hat{\mu}_t(\xi)|$  decays exponentially fast for  $|\xi| \rightarrow \infty$ . Together with the continuity of  $\xi \mapsto \Re(A(\xi))$  finiteness of the moments  $\int_{\mathbb{R}^d} |\xi|^n |\hat{\mu}_t(\xi)| d\xi < \infty$  for every  $n \in \mathbb{N}$  follows. The assertion now follows from Sato (1999, Proposition 28.1).  $\square$

Proposition 3.6 shows that the existence of a Sobolev index indicates the smoothness of the distribution of the process. Together with Proposition 6.6, the assertion establishes ties between the smoothness of the distribution and path properties of the process, see the comments below Remark 6.7.

Before proving that the Gårding condition on the symbol entails a Gårding inequality of the associated bilinear form, we derive an elementary inequality:

For  $C_1 > 0$ ,  $C_2 \geq 0$ ,  $0 \leq \beta < \alpha$  and  $0 < C_3 < C_1$  there exists a constant  $C_4 > 0$  such that

$$C_1 x^\alpha - C_2 x^\beta \geq C_3 x^\alpha - C_4 \quad \text{for all } x \geq 0. \quad (14)$$

To show inequality (14), it is enough to realize that for given constants  $C_1, C_2, C_3, \alpha$  and  $\beta$  as above, the point  $x_0 = \left( \frac{\beta C_2}{\alpha(C_1 - C_3)} \right)^{1/(\alpha - \beta)}$  is a global minimum of the function  $f(x) := (C_1 - C_3)x^\alpha - C_2 x^\beta$  on  $\mathbb{R}_{\geq 0}$ .

**Lemma 3.7.** Let  $A \in S_\alpha^0$ . If there exist constants  $C_2 > 0$ ,  $C_3 \geq 0$  and  $0 \leq \beta < \alpha$  with

$$\Re(A(\xi)) \geq C_2 |\xi|^\alpha - C_3 (1 + |\xi|^2)^{\beta/2} \quad (\xi \in \mathbb{R}^d),$$

then the corresponding bilinear form satisfies a Gårding inequality with respect to  $H^{\alpha/2}(\mathbb{R}^d) \hookrightarrow L^2(\mathbb{R}^d)$ , i.e. there exist constants  $c_2 > 0$  and  $c_3 \geq 0$  with

$$\Re(a(u, u)) \geq c_2 \|u\|_{\alpha/2}^2 - c_3 \|u\|_{L^2}^2.$$

*Proof.* For  $u \in S(\mathbb{R}^d)$  we have

$$\Re(a(u, u)) \geq \int \left( C_2 |\xi|^{2\alpha} - C_3 (1 + |\xi|^2)^\beta \right) |\hat{u}(\xi)|^2 d\xi.$$

Furthermore we have  $(1 + |x|^2)^\beta \leq 2^\beta (1 + |x|^{2\beta})$ , since for  $f(x) = (1 + |x|^{2\beta})$  and  $g(x) = (1 + |x|^2)^\beta$  we get

$$\frac{2^\beta f(x)}{g(x)} = \frac{2^\beta}{(1 + |x|^2)^\beta} + \left( \frac{2|x|^2}{1 + |x|^2} \right)^\beta.$$

The first summand is bigger or equal to 1 if  $x \leq 1$ , whereas the second summand is bigger or equal to 1 if  $x \geq 1$ . As both summands are positive for  $x \geq 0$ , we have  $2^\beta f(x) \geq g(x)$  for all  $x \geq 0$ . Together with inequality (14) this yields

$$C_2 |\xi|^{2\alpha} - C_3 (1 + |\xi|^2)^\beta \geq C_2 |\xi|^{2\alpha} - C'_3 (1 + |\xi|^{2\beta}) \geq c_2 (1 + |\xi|)^{2\alpha} - c_3$$

with a strictly positive constant  $c_2$  and  $C'_3, c_3 \geq 0$ , which yields the result.  $\square$

As argued to conclude Theorem 3.4, from the classical result on existence and uniqueness of solutions of parabolic differential equations, we obtain the following result.

**Theorem 3.8.** *Let  $\mathcal{A}$  be a PDO with symbol  $A$  and Sobolev index  $\alpha$  for some  $\alpha > 0$ . Then the parabolic equation*

$$\begin{aligned} \partial_t u + \mathcal{A}u &= f \\ u(0) &= g, \end{aligned} \tag{15}$$

for  $f \in L^2(0, T; H^{-\alpha/2}(\mathbb{R}^d))$  and  $g \in L^2(\mathbb{R}^d)$  has a unique weak solution  $u$  in the space  $W^1(0, T; H^{\alpha/2}(\mathbb{R}^d), L^2(\mathbb{R}^d))$ .

Moreover the solution  $u$  depends continuously on the data  $g$  and  $f$ . The proof of the classical theorem is based on a so-called Galerkin-approximation that yields a numerical scheme to calculate the solution approximately, namely a finite element scheme, see e.g. (Zeidler 1990, Theorem 23.A).

In light of Theorem 3.8, the Sobolev index appears as a measure of the degree of the smoothing effect of the related evolution problem. Under appropriate additional assumptions, a Feynman-Kac formula for weak solutions yields a stochastic representation. Thus, the Sobolev index represents a measure for the smoothing effect of the distribution of the Lévy process.

#### 4. SOBOLEV INDICES OF LÉVY PROCESSES

Let us observe that for two Lévy processes  $L^i$  with symbol  $A^i$  and Sobolev index  $\alpha_i$  for  $i = 1, 2$ , the sum  $L := L^1 + L^2$  is a Lévy process with symbol given by  $A := A^1 + A^2$ , and obviously the process has a Sobolev index that equals  $\max(\alpha_1, \alpha_2)$ .

**Example 4.1** (Lévy process with Brownian part).  *$\mathbb{R}^d$ -valued Lévy processes  $L$  with characteristics  $(b, \sigma, F)$  with a positive definite matrix  $\sigma$  have Sobolev index 2.*

*Proof.* Let us observe that

$$\Re(A(\xi)) = \frac{1}{2} \langle \xi, \sigma \xi \rangle + \int \left( 1 - \cos(\langle \xi, h(y) \rangle) \right) F(dy) \geq \frac{1}{2} \langle \xi, \sigma \xi \rangle.$$

Since the matrix  $\sigma$  is symmetric and positive definite  $\underline{\sigma}|\xi|^2 \leq \langle \xi, \sigma \xi \rangle$  for all  $\xi \in \mathbb{R}^d$ , where  $0 < \underline{\sigma}$  is the smallest eigenvalue of the matrix  $\sigma$ . As a consequence we have  $\underline{\sigma}|\xi|^2 \leq \Re(A(\xi))$ , i.e. the Gårding condition. Continuity follows immediately from inequality (5).  $\square$

**Example 4.2.** (Multivariate NIG-processes) Let  $L$  be an  $\mathbb{R}^d$ -valued NIG-process, i.e.

$$L_1 = (L_1^1, \dots, L_1^d) \sim \text{NIG}_d(\alpha, \beta, \delta, \mu, \Delta),$$

with parameters  $\alpha, \delta \in \mathbb{R}_{\geq 0}$ ,  $\beta, \mu \in \mathbb{R}^d$  and a symmetric positive definite matrix  $\Delta \in \mathbb{R}^{d \times d}$  with  $\alpha^2 > \langle \beta, \Delta \beta \rangle$ . Then the characteristic function of  $L_1$  in  $u \in \mathbb{R}^d$  is given by

$$E e^{i\langle u, L_1 \rangle} = \exp \left( i\langle u, \mu \rangle + \delta \left( \sqrt{\alpha^2 - \langle \beta, \Delta \beta \rangle} - \sqrt{\alpha^2 - \langle \beta + iu, \Delta(\beta + iu) \rangle} \right) \right),$$

where by  $\langle \cdot, \cdot \rangle$  we denote the product  $\langle z, z' \rangle = \sum_{j=1}^d z_j z'_j$  for  $z \in \mathbb{C}^d$ . Note that this is not the Hermitian scalar product. In (Barndorff-Nielsen 1977) multivariate NIG-distributions are derived as a subclass of multivariate GH-distributions via a mean variance mixture. We verify that  $\mathbb{R}^d$ -valued NIG-processes have Sobolev index 1.

*Proof.* Similar to the calculations in (Eberlein, Glau, and Papapantoleon 2010, Appendix B) for real-valued NIG-processes,

$$\begin{aligned} z &:= \alpha^2 - \langle \beta - iu, \Delta(\beta - iu) \rangle \\ &= \alpha^2 - \langle \beta, \Delta \beta \rangle + \langle u, \Delta u \rangle + i\langle \beta, \Delta u \rangle + i\langle u, \Delta \beta \rangle \end{aligned}$$

and  $\sqrt{z} = \sqrt{\frac{1}{2}(|z| + \Re(z))} + i \frac{\Im(z)}{|\Im(z)|} \sqrt{\frac{1}{2}(|z| - \Re(z))}$  it follows  $|z| \geq \alpha^2 - \langle \beta, \Delta \beta \rangle + \langle u, \Delta u \rangle > 0$  and

$$\begin{aligned} \Re(A(u)) &= -\delta \sqrt{\alpha^2 - \langle \beta, \Delta \beta \rangle} + \delta \Re(\sqrt{z}) \\ &= \frac{\delta}{\sqrt{2}} \sqrt{|z| + \Re(z)} - \delta \sqrt{\alpha^2 - \langle \beta, \Delta \beta \rangle} \\ &\geq \delta \sqrt{\alpha^2 - \langle \beta, \Delta \beta \rangle + \langle u, \Delta u \rangle} - \delta \sqrt{\alpha^2 - \langle \beta, \Delta \beta \rangle} \\ &\geq \delta \sqrt{\lambda_{\min}} |u| - \delta \sqrt{\alpha^2 - \langle \beta, \Delta \beta \rangle}, \end{aligned}$$

where  $\lambda_{\min}$  denotes the smallest eigenvalue of the matrix  $\Delta$ . Analogously it follows that  $|\Re(u)| \leq C_1(1 + |u|)$  and  $|\Im(u)| \leq C_2(1 + |u|)$  with positive constants  $C_1, C_2$ , which yields  $|A(u)| \leq C(1 + |u|)$  with a positive constant  $C$ .  $\square$

**Example 4.3** (Cauchy processes). Let  $L$  be a Cauchy process with values in  $\mathbb{R}^d$ , then the distribution  $\mu := P^{L_1}$  has the Lebesgue density

$$f(x) = c \frac{\Gamma((d+1)/2)}{\pi^{(d+1)/2}} (|x - \gamma|^2 + c^2)^{-(d+1)/2}$$

where  $\mu \in \mathbb{R}^d$  and  $c > 0$ , and its characteristic function is given in

$$\hat{\mu}(u) = e^{-c|u| + i\langle \gamma, u \rangle},$$

see (Sato 1999, Example 2.12). It follows immediately that the process has Sobolev index 1.

**Example 4.4** (Student- $t$  processes). Let  $L$  be a Lévy process such that the distribution of  $L_1$  is student- $t$  with parameters  $\mu \in \mathbb{R}$ ,  $f > 0$  and  $\delta > 0$  i.e.

$$P^{L_1}(dx) = \frac{\Gamma((f+1)/2)}{\sqrt{\pi\delta^2} \Gamma(f/2)} \left( 1 + \frac{x - \mu}{\delta^2} \right)^{-(f+1)/2}.$$

This generalisation of the student- $t$  distribution is studied in (Eberlein and Hammerstein 2004), where it appears as limit of GH distributions for parameters  $\alpha, \beta \downarrow 0$  with negative  $\lambda$ .

We show that  $L$  has Sobolev index 1.

*Proof.* The characteristic function  $\hat{\mu}$  of the student- $t$  distribution reads as follows

$$\hat{\mu}(u) = \left(\frac{f}{4}\right)^{f/4} \frac{2K_{-f/4}(\sqrt{f}|u|)}{\Gamma(f/2)} |u|^{f/4} e^{i\mu u}.$$

with  $\delta := f/4$  and  $c := \log \{(f/4)^{f/4}/\Gamma(f/2)\}$ , compare (Eberlein and Hammerstein 2004). We obtain the following representation of the associated symbol,

$$A(u) = -c - \log \left\{ K_{-\delta}(2\sqrt{\delta}|u|) \right\} - \log \left\{ |u|^{2\delta} \right\} + i\mu u. \quad (16)$$

Since the mapping  $u \mapsto A(u)$  is continuous, it is enough to verify the continuity and Gårding inequality for a function that is asymptotically equivalent to  $A$ . To this aim we insert the asymptotic expansion of the Bessel function  $K_\lambda$ , see (Abramowitz and Stegun 1964, equation (9.7.2)).

$$K_\lambda(z) \sim \sqrt{\frac{\pi}{2z}} e^{-z} \left\{ 1 + \frac{\mu-1}{8z} + \frac{(\mu-1)(\mu-9)}{2!(8z)^2} + \frac{(\mu-1)(\mu-9)(\mu-25)}{3!(8z)^3} + \dots \right\}$$

for  $|\arg z| < \frac{3}{2}\pi$  and  $|z| \rightarrow \infty$  with  $\mu = 4\lambda^2$  with the usual notation  $f(x) \sim g(x)$  for  $|x| \rightarrow \infty$  if  $\frac{f(x)}{g(x)} \rightarrow 1$  for  $|x| \rightarrow \infty$ .

In particular

$$K_\lambda(z) \sim \sqrt{\frac{\pi}{2z}} e^{-z} =: g(z) \quad \text{for } z \text{ real and } z \rightarrow \infty$$

and  $K_\lambda(z) \rightarrow 0$  as well as  $g(z) \rightarrow 0$  for  $z \rightarrow \infty$ . It follows

$$\frac{\log(K_\lambda(z))}{\log(g(z))} \sim \frac{g'(z)}{K'_\lambda(z)} \quad \text{with } K'_\lambda(z) = \frac{\lambda}{z} K_\lambda(z) - K_{\lambda+1}(z),$$

compare p. 79 equation (4) in (Watson 1922). We conclude

$$\frac{\log(K_{-\delta}(z))}{\log(g(z))} \sim \frac{-g(z) - \frac{1}{2z}g(z)}{\frac{-\delta}{z}K_{-\delta}(z) - K_{-\delta+1}(z)} \sim \frac{g(z)}{K_{-\delta}(z)} \sim 1.$$

Therefore

$$\log \left\{ K_{-\delta}(2\sqrt{\delta}|u|) \right\} \sim \log \left\{ \sqrt{\frac{\pi}{4\sqrt{\delta}|u|}} \right\} - 2\sqrt{\delta}|u| + \log \left\{ 1 + \left| O\left(\frac{1}{|u|}\right) \right| \right\} \quad (17)$$

for  $|u| \rightarrow \infty$ , where  $O$  denotes Landau's symbol, i.e. we write  $f(x) = O(g(x))$  for  $|x| \rightarrow \infty$  if there exists constants  $M, N$  s.t.  $\frac{|f(x)|}{|g(x)|} \leq M$  for all  $|x| > N$ . Inserting equation (17) in equation

(16) we obtain for the real part of the symbol

$$\begin{aligned}\Re(A(u)) &\sim -c - \log \left\{ \sqrt{\frac{\pi}{4\sqrt{\delta}|u|}} \right\} + 2\sqrt{\delta}|u| \\ &\quad - \log \left\{ 1 + \left| O\left(\frac{1}{|u|}\right) \right| \right\} - 2\delta \log |u|.\end{aligned}$$

From the boundedness of the term  $\log \left\{ 1 + \left| O\left(\frac{1}{|u|}\right) \right| \right\}$  for  $|u| \rightarrow \infty$  we further get

$$\begin{aligned}\Re(A(u)) &\sim 2\sqrt{\delta}|u| - (1 + 2\delta) \log |u| \\ &\geq 2\sqrt{\delta}|u| - (1 + 2\delta) \sqrt{|u|},\end{aligned}$$

since  $|\log |u|| \leq \sqrt{|u|}$ . This shows the Gårding-condition and moreover that  $|\Re(A(u))| \leq c|u|$  for some constant  $c \geq 0$ . Furthermore the imaginary part equals  $\Im(A(u)) = \mu u$ , hence the continuity condition is also satisfied.  $\square$

#### 4.1. Sobolev index for Lévy processes with absolutely continuous Lévy measure.

In this subsection, we study the Sobolev index for real-valued Lévy processes without Brownian part whose Lévy measure has a Lebesgue density. If the process has no Brownian part, the Gårding condition only depends on the real part of the integral  $\int (e^{-iux} - 1 + ih(x)u)F(dx)$ , which translates to properties of the symmetric part of the Lévy measure.

Let  $A$  be the symbol of a real-valued Lévy process that is a special semimartingale  $L$  with operator  $\mathcal{A}$ . Let  $(b, 0, F)$  be the characteristic triplet of  $L$  w.r.t.  $h(x) = x$ . Furthermore assume  $F(dx) = f(x)dx$  for the Lévy measure  $F$ . We denote by  $f_s$  the symmetric and by  $f_{as}$  the antisymmetric part of the density function  $f$ , i.e.  $f_s(x) = \frac{1}{2}(f(x) + f(-x))$  and  $f_{as}(x) = f_s(x) - f_s(-x)$  for every  $x \in \mathbb{R}$ .

For every  $u \in \mathbb{R}$  we define

$$\begin{aligned}A^{f_s}(u) &:= - \int (e^{-iux} - 1 + iux) f_s(x) dx = - \int (\cos(ux) - 1) f_s(x) dx, \\ A^{f_{as}}(u) &:= - \int (e^{-iux} - 1 + iux) f_{as}(x) dx = i \int (\sin(ux) - ux) f_{as}(x) dx, \\ A^f(u) &:= - \int (e^{-iux} - 1 + iux) f(x) dx = A^{f_s}(u) + A^{f_{as}}(u).\end{aligned}$$

Note the following equalities,

$$\Re(A^f) = A^{f_s}, \quad i\Im(A^f) = A^{f_{as}}, \quad (18)$$

$$\Re(A(u)) = \frac{1}{2}\langle u, \sigma u \rangle + A^{f_s}(u), \quad (19)$$

$$\Im(A(u)) = \langle u, b \rangle - iA^{f_{as}}(u) = \langle u, b \rangle + \int (\sin(ux) - ux) f_{as}(x) dx. \quad (20)$$

Let us further notice the following elementary assertion.

**Lemma 4.5.** *If  $F$  is a nonnegative measure, absolutely continuous with respect to the Lebesgue measure, it has a nonnegative density  $f \geq 0$  and  $|f_{as}| \leq f_s$ .*

*Proof.* If  $F(dx) = f(x)dx$  with a nonnegative measure  $F$ , then  $F(dx) = |f(x)|dx$  hence w.l.g.  $f(x) \geq 0$ .

Thus we have  $f_s \geq 0$ , since otherwise there would exist a number  $y \in \mathbb{R}$  with  $f_s(y) < 0$  and  $f_s(-y) = f_s(y) < 0$ . However, since  $f_{as}(y) \leq 0$  or  $f_{as}(-y) \leq 0$  we would get  $f(y) < 0$  or  $f(-y) < 0$  i.e. a contradiction.

Furthermore we have  $|f_{as}| \leq f_s$ , since otherwise there would exist a number  $y \in \mathbb{R}$  with  $-f_{as}(y) > f_s(y)$  i.e.  $f(y) < 0$  or  $f_{as}(y) > f_s(y)$ , from where we would get  $f(-y) < 0$ .  $\square$

In the following proposition we derive the Sobolev index for Lévy processes without Brownian part from the behaviour of the Lévy measure  $F$  around the origin. It uses a rather technical lemma that is provided in appendix A.

**Proposition 4.6.** *Let  $L$  be a real-valued Lévy process and a special semimartingale with characteristic triplet  $(b, 0, F)$  with respect to the truncation function  $h(x) = x$ .*

*Let*

$$f_s(x) = \frac{C}{|x|^{1+Y}} + g(x) \quad \text{with } g(x) = O\left(\frac{1}{|x|^{1+Y-\delta}}\right) \text{ for } x \rightarrow 0 \quad (21)$$

*with  $0 < \delta$ . In the following cases, the Lévy process  $L$  has Sobolev index  $Y$ .*

a) *Let  $0 < Y < 1$  and*

$$f_{as}(x) = O\left(\frac{1}{|x|^\alpha}\right) \quad \text{for } x \rightarrow 0$$

*with  $\alpha \leq 1 + Y$ ,  $\int |x f_{as}(x)| dx < \infty$ , and moreover  $b = \int x F(dx)$ .*

b) *Let  $Y = 1$  and*

$$f_{as}(x) = O\left(\frac{1}{|x|^\alpha}\right) \quad \text{for } x \rightarrow 0$$

*with  $\alpha < 1 + Y = 2$ .*

c) *Let  $1 < Y < 2$ .*

*Proof.* In each of the three cases, according to part b) of Lemma A.1, the Gårding condition follows directly from

$$\Re(A(u)) = \Re(A^f(u)) = \Re(A^{f_s}(u)) \geq C|u|^Y - C_1(1 + |u|^{Y'})$$

with  $C > 0$  and  $C_1 \geq 0$  and  $0 < Y' < Y$ .

Splitting  $A$  in its real and its imaginary part, assertion a) and d) of Lemma A.1 yield the continuity condition in case a) with index  $Y$ , since the assumption  $f_{as}(x) = O\left(\frac{1}{|x|^\alpha}\right)$  for some  $\alpha \in (0, 1 + Y]$  implies  $f_{as}(x) = O\left(\frac{1}{|x|^{1+Y'}}\right)$ .

In order to verify the continuity conditions for b) and c), we first notice that

$$|A(u)| \leq |A^{f_s}(u)| + \left| \Im(A^f(u)) \right| + |b||u| \leq C_2(1 + |u|^Y) + |A^{f_{as}}(u)| + |b||u|$$

follows from Lemma A.1 a).

Concerning case b), we notice that the assumption on  $f_{as}$  implies  $f_{as}(x) = O\left(\frac{1}{|x|^{1+Y'}}\right)$  with some  $0 < Y' < 1$ . Lemma A.1 c) yields  $|A^{f_{as}}(u)| \leq C_3(1 + |u|)$ . Together with Lemma A.1 a) this yields

$$|A(u)| \leq C_2(1 + |u|) + C'_2(1 + |u|^{1-\delta}) + C_3(1 + |u|) + |b||u| \leq C(1 + |u|)$$

with nonnegative constants  $C_2, C'_2, C_3$  and  $C$ . In other words we have shown the continuity condition for case b).

Finally, to verify the continuity condition for  $1 < Y < 2$ , let us notice that from Lemma 4.5 we know  $|f_{as}| \leq f_s$  so that

$$|f_{as}(x)| = O\left(\frac{1}{|x|^{1+Y}}\right)$$

for  $|x| \rightarrow 0$ . Due to Lemma A.1 c) we have  $|A^{f_{as}}(u)| \leq C_3 (1 + |u|^Y)$  and altogether we obtain

$$|A(u)| \leq C_2 (1 + |u|^Y) + C_3 (1 + |u|^Y) + |b||u| \leq C' (1 + |u|^Y) .$$

□

**Example 4.7.** *Generalised Hyperbolic (GH) processes have Sobolev index 1.*

*Proof.* The Lévy measure  $F^{GH}$  of a GH process has a Lebesgue density  $F^{GH}(dx) = f^{GH}(x) dx$  with

$$f^{GH}(x) = C_1 \frac{1}{x^2} + C_2 \frac{1}{|x|} + C_3 \frac{1}{x} + \frac{o(|x|)}{x^2}$$

with  $\frac{o(|x|)}{|x|} \rightarrow 0$  for  $|x| \rightarrow 0$ , see Raible (2000, Proposition 2.18). Hence the symmetric part of  $f^{GH}$  is of the form

$$f_s^{GH}(x) = \frac{C}{|x|^2} + O\left(\frac{1}{|x|}\right) \quad \text{for } x \rightarrow 0,$$

and the antisymmetric part is of the form

$$f_{as}^{GH}(x) = O\left(\frac{1}{|x|}\right) \quad \text{for } x \rightarrow 0.$$

The assertion follows from part b) of Theorem 4.6. □

**Example 4.8.** *A CGMY Lévy process with parameters  $C, G, M > 0$  and  $Y < 2$ , is a Lévy process that has no Brownian part and its Lévy measure  $F^{CGMY}$  is given by its Lebesgue density*

$$f^{CGMY}(x) = \begin{cases} \frac{C}{|x|^{1+Y}} e^{Gx} & \text{for } x < 0 \\ \frac{C}{|x|^{1+Y}} e^{-Mx} & \text{for } x \geq 0, \end{cases}$$

compare (Carr, Geman, Madan, and Yor 2002).

- (i) *A CGMY Lévy process with parameters  $C, G, M > 0$  and  $Y \in (0, 1)$  and characteristics  $(Y(M^{Y-1} - G^{Y-1}), 0, F^{CGMY})$  with respect to the truncation function  $h(x) = x$  has Sobolev index  $Y$ .*
- (ii) *A CGMY Lévy process with parameters  $C, G, M > 0$  and  $Y \in [1, 2)$  has Sobolev index  $Y$ .*

*Proof.* For  $Y \in (0, 1)$ , the assertion follows immediately from the explicit formula of the characteristic exponent of the distribution. Namely for a CGMY process  $L$  with characteristics  $(0, 0, F^{CGMY})$  w.r.t. the truncation function  $h(x) = x$  we have

$$\begin{aligned} \log(E e^{iuL_1}) &= C\Gamma(-Y)\{(M - iu)^Y - M^Y + Y(M^{Y-1} - G^{Y-1})iu \\ &\quad + (G + iu)^Y - G^Y\}, \end{aligned} \tag{22}$$

where  $\Gamma$  denotes the analytic extension of the Gamma function, see (Poirot and Tankov 2006).

For  $Y \geq 1$  no explicit formula is available, we therefore examine the density of the Lévy measure. The following decomposition in a symmetric and an antisymmetric part of the density is valid for any  $Y \in (0, 2)$ ,

$$\begin{aligned} f_s^{\text{CGMY}}(x) &= \frac{C}{2} \frac{e^{-G|x|} + e^{-M|x|}}{|x|^{1+Y}} = \frac{C}{|x|^{1+Y}} - C \frac{2 - e^{-G|x|} - e^{-M|x|}}{2|x|^{1+Y}} \\ &= \frac{C}{|x|^{1+Y}} + O\left(\frac{1}{|x|^Y}\right) \quad \text{for } |x| \rightarrow 0. \end{aligned}$$

Furthermore we have

$$|f_{as}^{\text{CGMY}}(x)| = \frac{C}{2} \frac{|e^{-G|x|} - e^{-M|x|}|}{|x|^{1+Y}} = O\left(\frac{1}{|x|^Y}\right) \quad \text{for } |x| \rightarrow 0.$$

For  $Y \geq 1$  we obtain from Proposition 4.6 b) and c) that  $L$  has Sobolev index  $Y$ .  $\square$

We conclude this section with the following observation.

**Remark 4.9.** *Variance gamma (VG) processes are CGMY processes with parameter  $Y = 0$ , and they do not have a Sobolev index.*

## 5. SOBOLEV INDEX OF $\alpha$ -SEMI-STABLE LÉVY PROCESSES

Remember that for an  $\alpha$ -semi-stable Lévy process  $L$  there exists a deterministic function  $t \mapsto c(t)$  and  $a > 1$  such that  $(L_{at})_{t \geq 0}$  coincides with  $(a^{1/\alpha}L_t + c(t))_{t \geq 0}$  in distribution, compare Section 13 in (Sato 1999).

The symbol of a generic real-valued strictly  $\alpha$ -stable Lévy process is of the form  $A(u) = c|u|^\alpha$  with a constant  $c > 0$ , see (Sato 1999, Theorem 14.9). In this case the Lévy process obviously has Sobolev index  $\alpha$ .

In this section we show that any  $\alpha$ -semi-stable Lévy process with  $1 < \alpha \leq 2$  has Sobolev index  $\alpha$ . For  $\alpha$ -semi-stable Lévy processes with  $0 < \alpha < 1$  we give additional sufficient conditions under which the processes have Sobolev index  $\alpha$ . Additionally, it turns out that any real-valued strictly  $\alpha$ -stable Lévy process has Sobolev index  $\alpha$ .

Let us give a definition of (semi) stability and  $\alpha$ -(semi) stability of Lévy processes in terms of the symbol of the process according to Definition 13.1, Proposition 13.5, Definition 13.16, and Theorem 13.11 in (Sato 1999).

**Definition 5.1.** *A Lévy process with symbol  $A$  is called (semi) stable if for any  $0 < a \neq 1$  (for some  $0 < a \neq 1$ ) there exists a constant  $b > 0$  and a vector  $c \in \mathbb{R}^d$  with*

$$aA(u) = A(bu) + i\langle c, u \rangle \quad \text{for all } u \in \mathbb{R}^d. \quad (23)$$

*A Lévy process is called  $\alpha$ -semi-stable, if it is semi-stable and if*

$$a = b^\alpha \quad \text{for all } a \in \Gamma \quad (24)$$

*with  $\Gamma := \{a > 0 \mid \exists b > 0, c \in \mathbb{R}^d \text{ s.t. (23) is satisfied with } a, b \text{ and } c\}$ . Accordingly, a Lévy process is called  $\alpha$ -stable, if it is stable and if*

$$a = b^\alpha \quad \text{for all } a \in \Gamma = (0, \infty). \quad (25)$$

*If  $c = 0$  in equality (23), the process is called strictly semi-stable, strictly stable resp. strictly  $\alpha$ -(semi) stable.*

From Definition 13.16, Theorem 13.15 and from Theorem 14.1 in (Sato 1999) we obtain the following remark.

**Remark 5.2.** *Let  $L$  be an  $\alpha$ -semi-stable Lévy process with characteristic triplet  $(b, \sigma, F)$ , with  $\sigma \neq 0$  or  $F \neq 0$ .*

- a) *We have  $0 < \alpha \leq 2$ .*
- b) *We have  $\alpha = 2$  iff  $\sigma \neq 0$  and  $F \equiv 0$ .*

Before focusing on the Sobolev index for  $\alpha$ -semi stable Lévy processes, we briefly discuss the notion of (non-)degeneracy of Lévy processes.

According to Definition 24.16 and 24.18 in (Sato 1999), an  $\mathbb{R}^d$ -valued Lévy process  $L$  is called *degenerate*, if  $P^{L_t}$  is degenerate for any (or equivalently for some)  $t > 0$ , i.e.

$$S_{P^{L_t}} = \{x \in \mathbb{R}^d \mid P^{L_t}(G) > 0 \text{ for every open subset } G \in \mathbb{R}^d \text{ with } x \in G\}$$

is contained in some affine subspace of  $\mathbb{R}^d$ , i.e.

$$S_{P^{L_t}} \subset y + V$$

for some  $y \in \mathbb{R}^d$  and some linear  $(d-1)$ -dimensional subspace of  $\mathbb{R}^d$ .

A Lévy process that is not degenerate is said to be *nondegenerate*. Note that the definition of degeneracy implies that non-constant real-valued Lévy processes are nondegenerate. Proposition 24.17 (ii) shows that an  $\mathbb{R}^d$ -valued Lévy process is nondegenerate, if its Lévy measure is nondegenerate or if  $\sigma(\mathbb{R}^d) = \{\sigma x \mid x \in \mathbb{R}^d\}$  is not contained in a  $(d-1)$ -dimensional linear subspace of  $\mathbb{R}^d$ .

From (Sato 1999, Proposition 24.20), the following relation between the Sobolev index and  $\alpha$ -stability can be deduced.

**Proposition 5.3.** *Every nondegenerate  $\alpha$ -semi-stable Lévy process satisfies the Gårding condition with index  $\alpha$ .*

*Proof.* The assertion follows directly from Proposition 24.20 in (Sato 1999). □

In view of Proposition 5.3, it is enough to study the continuity condition in the sequel. The following proposition characterises the Sobolev index for  $\alpha$ -semi-stable Lévy processes with  $\alpha \neq 1$  and for real-valued 1-stable Lévy processes.

**Proposition 5.4.** *Let  $L$  be a nondegenerate  $\alpha$ -semi-stable Lévy process.*

- a) *If  $0 < \alpha < 1$ , then  $L$  has Sobolev index  $\alpha$  iff  $L$  is strictly  $\alpha$ -semi-stable.*
- b) *If  $1 < \alpha \leq 2$ , then  $L$  has Sobolev index  $\alpha$ .*
- c) *The symbol of a real-valued strictly  $\alpha$ -stable Lévy process with  $\alpha = 1$  and Sobolev index 1 is of the form*

$$A(u) = c|u| + i\tau u$$

*with  $c > 0$  and  $\tau \in \mathbb{R}$ .*

- d) *If the process  $L$  is real-valued and  $\alpha$ -stable with  $\alpha = 1$ , then  $L$  has Sobolev index 1 iff  $L$  is strictly 1-stable.*

*Proof.* In view of Remark 5.2 the assertion is obvious for  $\alpha = 2$ .

For  $0 < \alpha < 2$  with  $\alpha \neq 1$ , Proposition 14.9 in Sato (1999) shows that the symbol  $A = -\log(\hat{\mu})$  is of the form

$$A(u) = |u|^\alpha (\eta(u) + i\gamma_\alpha(u)) + i\langle c_\alpha, u \rangle$$

with  $c_\alpha \in \mathbb{R}^d$ ,  $u \mapsto \eta(u)$  nonnegative, continuous on  $\mathbb{R}^d \setminus \{0\}$  and  $\eta(bu) = \eta(u)$  for all  $u \in \mathbb{R}^d$ , and  $\gamma_\alpha$  real-valued, continuous on  $\mathbb{R}^d \setminus \{0\}$  with  $\gamma_\alpha(bu) = \gamma_\alpha(u)$  for all  $u \in \mathbb{R}^d$  with  $b = a^{1/\alpha} > 1$ . Basic arguments show that the mappings  $u \mapsto \eta(u)$  and  $u \mapsto \gamma_\alpha(u)$  are bounded. We therefore have  $\Re(A(u)) = |u|^\alpha \eta(u)$ , where  $\eta$  is bounded, and hence  $|\Re(A(u))| \leq C|u|^\alpha$ . In view of Proposition 5.3 it remains to derive an adequate upper bound of the imaginary part.

For  $0 < \alpha < 2$ ,  $\alpha \neq 1$  we have

$$\Im(A(u)) = |u|^\alpha \gamma_\alpha(u) + \langle c_\alpha, u \rangle$$

with the bounded function  $\gamma_\alpha$ .

For  $0 < \alpha < 1$ , this shows that  $|\Im(A(u))| \leq C(1 + |u|^\alpha)$  iff  $c_\alpha = 0$ . According to (Sato 1999, Theorem 14.7 (i)) the latter is the case if and only if the distribution resp. the Lévy process is strictly  $\alpha$ -semi-stable.

For  $1 < \alpha < 2$ , due to

$$|\langle c_\alpha, u \rangle| \leq |c_\alpha| |u| \leq |c_\alpha| (1 + |u|^\alpha),$$

we obtain  $|\Im(A(u))| \leq C(1 + |u|^\alpha)$  without further restrictions.

Assertion c) and d) for  $\alpha = 1$  are a direct consequence of Theorem 14.15, equation (14.25) in (Sato 1999), that states that the symbol of a real-valued non-trivial (i.e. a non-constant) 1-stable Lévy process is the form

$$A(u) = c|u| \left( 1 - i\beta \frac{2}{\pi} \frac{u}{|u|} \log |u| \right) + i\tau u$$

with  $c > 0$ ,  $\beta \in [-1, 1]$  and  $\tau \in \mathbb{R}$ . From this representation of the symbol we can read that  $L$  is strictly 1-stable iff  $\beta = 0$ . The representation given in assertion c) follows as well.  $\square$

## 6. CONNECTIONS WITH THE BLUMENTHAL-GETTOOR INDEX

The index  $\beta$ , called Blumenthal-Gettoor index, quantifies the intensity of small jumps of a Lévy process. It is defined for every Lévy process, whereas not every Lévy process has a Sobolev index. In this section we show for real-valued Lévy processes that if they have a Sobolev index  $Y < 2$ , then this index is bigger or equal to the Blumenthal-Gettoor index.

The following definition of the Blumenthal-Gettoor index is taken from Sato (1999, p. 362).

**Definition 6.1.** *Let  $L$  be a Lévy process with characteristics  $(b, c, F)$ . Then*

$$\beta := \inf \left\{ \alpha > 0 \mid \int_{[-1,1]} |x|^\alpha F(dx) < \infty \right\}$$

*is called the Blumenthal-Gettoor index of the process.*

It is well known that the Blumenthal-Gettoor index is related to path properties of the Lévy process. Theorem 21.9 in Sato (1999) shows the following relationship between the Blumenthal-Gettoor index and the variation of the paths of the Lévy process.

**Proposition 6.2.** *Let  $L$  be a Lévy process without Brownian part with characteristics  $(b, 0, F)$  and Blumenthal-Gettoor index  $\beta$ .*

- (a) *If  $\beta < 1$ , then  $P$ -a.e. path of  $L$  is of bounded variation on  $(0, t]$  for every  $t > 0$ .*
- (b) *If  $\beta > 1$ , then  $P$ -a.e. path of  $L$  is of unbounded variation on  $(0, t]$  for every  $t > 0$ .*
- (c) *If  $\beta = 1$ , then we have the following two cases.*

- (c1) If  $\int_{-1}^1 |x|F(dx) < \infty$ , then  $P$ -a.e. path of  $L$  is of bounded variation on  $(0, t]$  for every  $t > 0$ .  
(c2) If  $\int_{-1}^1 |x|F(dx) = \infty$ , then  $P$ -a.e. path of  $L$  is of unbounded variation on  $(0, t]$  for every  $t > 0$ .

In Hudson and Mason (1976) and the references therein this assertion is generalised for the so-called  $p$ -variation. In Woerner (2007) a *normed*  $p$ -variation is introduced and for the time-changed processes a relation to the Blumenthal-Gettoor index is derived.

In particular, for Lévy processes  $L$  it is shown under some assumptions on the characteristic triplet, that the normed  $p$ -variation for  $0 < p < \beta$  with  $p \neq \beta - 1$ , exists on a finite time interval  $[0, T]$ ,

$$\Delta_n^{1-p/\beta} \sum_{i=1}^{n-1} |L_{(i+1)\Delta_n} - L_{i\Delta_n}|^p \xrightarrow{P} V_p(L), \quad (26)$$

where  $V_p(L)$  is a finite number,  $\beta$  is the Blumenthal-Gettoor index, and for  $n \rightarrow \infty$  the partition of the time interval  $[0, T]$  refines uniformly,  $\Delta_n \downarrow 0$ , see (Woerner 2007, Theorem 1) and (Woerner 2003, Corollary 1). The additional assumptions made therein, concern the Lévy-measure, that is assumed to have a density with a certain Taylor expansion around the origin, and a special choice of the drift.

In order to compare the Sobolev index with the Blumenthal-Gettoor index, we introduce another index  $\gamma$  that, similar to the Blumenthal-Gettoor index, quantifies the intensity of small jumps of the process.

**Lemma 6.3.** *Let  $L$  be a Lévy process with characteristics  $(b, c, F)$  and with Blumenthal-Gettoor index  $\beta$ . We define the index*

$$\gamma := \sup \left\{ \alpha > 0 \mid \liminf_{r \downarrow 0} r^{\alpha-2} \int_{[-r, r]} |x|^2 F(dx) > 0 \right\}.$$

We have

$$\beta \geq \gamma.$$

*Proof.* For  $0 < \alpha < 2$  and  $0 < r < 1$  we have  $r^{\alpha-2} \int_{-r}^r |x|^2 F(dx) \leq \int_{-r}^r |x|^\alpha F(dx)$ , since

$$\int_{-r}^r |x|^2 r^{\alpha-2} F(dx) \leq \int_{-r}^r |x|^\alpha F(dx).$$

If

$$\liminf_{r \downarrow 0} r^{\alpha-2} \int_{-r}^r |x|^2 F(dx) > 0,$$

then there exists a constant  $C > 0$  with  $\int_{-r}^r |x|^\alpha F(dx) > C$  for all  $0 < r$  smaller than some  $\epsilon > 0$ . Hence  $\int_{-1}^1 |x|^\alpha F(dx) = \infty$  follows from  $F(\{0\}) = 0$ . This means that for every  $\alpha < \gamma$  we have  $\alpha \leq \beta$  whence  $\gamma \leq \beta$ .  $\square$

The index  $\gamma$  quantifies the intensity of small jumps of the Lévy process, hence it is also a measure for the regularity of the underlying distribution. Sato (1999, Proposition 28.3) shows the following remark.

**Remark 6.4.** *If we have  $0 < \gamma < 2$ , then the distribution  $\mu_1 = P^{L_1}$  does possess a smooth Lebesgue density.*

The rest of the section is dedicated to the relation between the index  $\gamma$ , the Blumenthal Gettoor-index, and the Sobolev index of a real-valued Lévy process. We therefore restrict ourselves to Lévy processes that take values in  $\mathbb{R}$ .

**Proposition 6.5.** *Let  $L$  be a real-valued Lévy process with characteristic triplet  $(b, 0, F)$ . If the index  $\gamma$  satisfies  $\gamma \in (0, 2)$ , then the symbol  $A$  of  $L$  satisfies a Gårding-condition for any index  $\alpha < \gamma$ .*

*Proof.* Let us write down the real part of the symbol,

$$\Re(A(\xi)) = \int (1 - \cos(\xi y)) F(dy).$$

As in the proof of Sato (1999, Proposition 28.3) we further conclude

$$\begin{aligned} & \int (1 - \cos(vy)) F(dy) \\ &= 2 \int_{|v||y| \leq \pi} \sin^2\left(\frac{vy}{2}\right) F(dy) + 2 \int_{|v||y| > \pi} \sin^2\left(\frac{vy}{2}\right) F(dy) \\ &\geq 2 \int_{|v||y| \leq \pi} \frac{2}{\pi^2} v^2 y^2 F(dy) + 2 \int_{|v||y| > \pi} \sin^2\left(\frac{vy}{2}\right) F(dy) \\ &\geq c' \int_{|v||y| \leq \pi} v^2 y^2 F(dy) \end{aligned} \tag{27}$$

for a positive constant  $c'$ . On the other hand, for  $\alpha < \gamma$  we have

$$\liminf_{r \downarrow 0} r^{\alpha-2} \int_{[-r, r]} x^2 F(dx) > 0,$$

hence there exists a constant  $c_1 > 0$  and  $\epsilon > 0$ , such that

$$\int_{[-r, r]} x^2 F(dx) \geq c_1 r^{2-\alpha} \quad \text{for every } r < \epsilon.$$

That is, for every  $\alpha < \gamma$  there exists an  $N > 0$  and a constant  $c_c > 0$  with

$$\int_{|v||y| \leq \pi} v^2 y^2 F(dy) \geq c_c |v|^\alpha \quad \text{for all } v \text{ with } |v| > N.$$

Altogether, we have

$$\Re(A(\xi)) \geq c |\xi|^\alpha \mathbf{1}_{|\xi| > N} \geq c |\xi|^\alpha - c_2$$

with  $c > 0$  and  $c_2 = c|N|^\alpha$ . □

**Proposition 6.6.** *Let  $L$  be a real-valued Lévy process with characteristics  $(b, 0, F)$ , and s.t. its symbol satisfies the Gårding-condition with  $0 < Y < 2$ . Then*

$$\int_{-1}^1 |x|^\alpha F(dx) = \infty \quad \text{for all } \alpha < Y < 2.$$

*In particular the Blumenthal-Gettoor index  $\beta$  of the process is bigger or equal to  $Y$  ( $\beta \geq Y$ ).*

*Proof.* From the assumption we know that there exist constants  $C_1 > 0$  and  $C_2 \geq 0$  and indexes  $0 < Y' < Y < 2$  with

$$\int (1 - \cos(ux)) F(dx) \geq C_1 |u|^Y - C_2 (1 + |u|^{Y'}) .$$

Thus for every  $\epsilon > 0$ , the inequality

$$\int_{-\epsilon}^{\epsilon} (1 - \cos(ux)) F(dx) \geq C_1 |u|^Y - C_2 |u|^{Y'} - C_\epsilon$$

holds for  $C_\epsilon = C_2 + 2F((-\epsilon, \epsilon)^c)$ . Since for every  $0 < \alpha < 2$  there exists a constant  $C(\alpha) > 0$  with  $1 - \cos(y) \leq C(\alpha)|y|^\alpha$  for all  $y \in \mathbb{R}$ , we are able to conclude for any fixed  $\epsilon > 0$  that

$$C_1 |u|^Y - C_2 |u|^{Y'} - C_\epsilon \leq \int_{-\epsilon}^{\epsilon} (1 - \cos(ux)) F(dx) \leq C(\alpha) \int_{-\epsilon}^{\epsilon} |ux|^\alpha F(dx)$$

for all  $u \in \mathbb{R}$ , resp.

$$\frac{C_1}{C(\alpha)} |u|^{Y-\alpha} - \frac{C_2}{C(\alpha)} |u|^{Y'-\alpha} - \frac{C_\epsilon}{C(\alpha)} |u|^{-\alpha} \leq \int_{-\epsilon}^{\epsilon} |x|^\alpha F(dx) \quad \text{for all } u \in \mathbb{R} \setminus \{0\} .$$

For every  $\epsilon > 0$  the left hand side of the inequality diverges for  $|u| \rightarrow \infty$ , if  $\alpha < Y$ . Thus we can conclude  $\infty = \int_{-\epsilon}^{\epsilon} |x|^\alpha F(dx)$  for every  $\epsilon > 0$ .  $\square$

Proposition 6.6 and Proposition 3.6 together yield the following result.

**Remark 6.7.** *The relation between both indexes, the Blumenthal-Gettoor and the Sobolev index, bridges the path properties and the distribution of the process. If the Lévy process has a Sobolev index, its distribution is smooth. Furthermore, its paths are of unbounded variation if the Sobolev index is bigger or equal to 1.*

This relation can be studied more extensively using a certain type of Feynman-Kac formula and results on  $p$ -variations of the process. (Woerner 2007) shows convergence in probability of the normed  $p$ -variation (26) under appropriate conditions on the Lévy process. This can be interpreted as a result on the intensity of oscillations of the paths of the process. On the other hand, Feynman-Kac formulas allow us to interpret the degree of smoothness of the solution of the PIDE as an effect that directly stems from the distribution. An appropriate Feynman-Kac formula that allows us to distinguish between different degrees of smoothing is given in (Glau 2010, Theorem IV.9).

To conclude, let us point out that the results in this article have an obvious extension to the case of time-inhomogeneous Lévy process when one requires continuity and Gårding condition *uniformly in time*. Moreover, for applications to option pricing, continuity and

Gårding condition are studied for an analytical extension of the symbol to a certain domain in the complex plane in the article (Eberlein and Glau 2011). For a more extensive study of multivariate processes, anisotropic Sobolev-Slobodeckii spaces are the appropriate spaces. Moreover, an extension of the framework to affine processes is a current research topic. This extension is not obvious, since the symbol of an affine process is affine in the state space, hence uniform bounds with respect to the Sobolev-Slobodeckii norms are not available.

#### APPENDIX A.

The following lemma relates the behaviour of the symbol  $A(u)$  for  $|u| \rightarrow \infty$  with the behaviour of the Lévy measure  $F$  around the origin.

Again, we use Landau's symbol  $O$  to indicate the asymptotic behaviour; here we look at the behaviour of a function around the origin. More precisely we write  $f(x) = O(g(x))$  for  $x \rightarrow 0$  if there exist positive constants  $M$  and  $N$  such that  $\frac{|f(x)|}{|g(x)|} \leq M$  for all  $|x| < 1/N$ .

As generally assumed in Section 4.1, let  $L$  be a real-valued Lévy process that is a special semimartingale with characteristic triplet  $(b, 0, F)$  w.r.t.  $h(x) = x$ . Furthermore assume  $F(dx) = f(x) dx$  for the Lévy measure  $F$  and we denote by  $f_s$  the symmetric and by  $f_{as}$  the antisymmetric part of the density function  $f$ .

**Lemma A.1.** *Let  $0 < Y < 2$ .*

a) *If  $f_s(x) = O\left(\frac{1}{|x|^{1+Y}}\right)$  for  $x \rightarrow 0$ , then there exists a constant  $C \geq 0$  with*

$$0 \leq \Re\left(A^f(u)\right) = A^{f_s}(u) \leq C(1 + |u|^Y) \quad \text{for all } u \in \mathbb{R}.$$

b) *If  $f_s(x) = \frac{C}{|x|^{1+Y}} + g(x)$  with  $g(x) = O\left(\frac{1}{|x|^{1+Y-\delta}}\right)$  for  $x \rightarrow 0$  with some  $0 < \delta$  and  $C > 0$ , then there exist constants  $C_1 > 0$ ,  $C_2 \geq 0$  and  $Y' \in (0, Y)$  such that*

$$\Re\left(A^f(u)\right) = A^{f_s}(u) \geq C_1|u|^Y - C_2(1 + |u|^{Y'}) \quad \text{for all } u \in \mathbb{R}.$$

c) *If  $f_{as}(x) = O\left(\frac{1}{|x|^{1+Y}}\right)$  for  $x \rightarrow 0$  with  $0 < Y$  and  $Y \neq 1$ , then there exist constants  $C$ ,  $C_1 \geq 0$  with*

$$\left|\Im\left(A^f(u)\right)\right| = \left|A^{f_{as}}(u)\right| \leq C(1 + |u| + |u|^Y) \leq C_1(1 + |u|^{\max[1, Y]})$$

*for every  $u \in \mathbb{R}$ .*

d) *Let  $f_{as}(x) = O\left(\frac{1}{|x|^{1+Y}}\right)$  for  $x \rightarrow 0$  with  $Y \in (0, 1)$  and assume  $\int |x|f(x) dx < \infty$  i.e. the paths of the process are a.s. of finite variation.*

*If  $L$  is a Lévy process with characteristic triplet  $(\int xF(dx), 0, F)$  w.r.t. the truncation function  $h(x) = x$ , then there exists a constant  $C \geq 0$  with*

$$\left|\Im(A(u))\right| \leq C(1 + |u|^Y) \quad \text{for all } u \in \mathbb{R}.$$

*Proof.* Proof of a): For every  $\epsilon > 0$  and arbitrary  $u \in \mathbb{R}$  we have

$$A^{f_s}(u) = \int_{-\epsilon}^{\epsilon} (1 - \cos(ux)) f_s(x) dx + \int_{(-\epsilon, \epsilon)^c} (1 - \cos(ux)) f_s(x) dx$$

with

$$0 \leq \int_{(-\epsilon, \epsilon)^c} (1 - \cos(ux)) f_s(x) dx \leq 2 \int_{(-\epsilon, \epsilon)^c} f_s(x) dx =: C(\epsilon).$$

If we choose  $\epsilon > 0$  small enough, we get

$$\begin{aligned} & \int_{-\epsilon}^{\epsilon} (1 - \cos(ux)) f_s(x) dx \\ & \leq C_1(\epsilon) \int_{-\epsilon}^{\epsilon} (1 - \cos(ux)) \frac{1}{|x|^{1+Y}} dx \\ & = C_1(\epsilon) |u|^Y \int_{-\epsilon|u|}^{\epsilon|u|} \frac{1 - \cos x}{|x|^{1+Y}} dx \\ & = 2C_1(\epsilon) |u|^Y \left( \int_0^1 \frac{1 - \cos x}{|x|^{1+Y}} dx + \int_1^{\epsilon|u|} \frac{1 - \cos x}{|x|^{1+Y}} dx \right) \end{aligned}$$

with a constant  $C_1(\epsilon) > 0$  only depending on  $\epsilon$ . Furthermore,

$$\int_0^1 \frac{1 - \cos x}{|x|^{1+Y}} dx \leq \frac{1}{2} \int_0^1 \frac{x^2}{|x|^{1+Y}} dx = \frac{1}{2} \int_0^1 x^{1-Y} dx = \frac{1}{2(2-Y)} < \infty,$$

since  $Y < 2$ . The second integral is negative for  $\epsilon|u| < 1$ , and for  $1 < \epsilon|u|$  we get

$$0 \leq \int_1^{\epsilon|u|} \frac{1 - \cos x}{|x|^{1+Y}} dx \leq \int_1^{\epsilon|u|} \frac{2}{|x|^{1+Y}} dx = \frac{2}{Y} \left( -(\epsilon|u|)^{-Y} + 1 \right) \leq \frac{2}{Y}.$$

So there exist  $\epsilon > 0$  and  $C_2(\epsilon) \geq 0$  with

$$\int_{-\epsilon}^{\epsilon} (1 - \cos(ux)) f_s(x) dx \leq C_2(\epsilon) |u|^Y. \quad (28)$$

For an appropriate choice of  $\epsilon$  we directly obtain the assertion of a).

Proof of b): The first equality of the assertion is given by (19). For every  $\epsilon > 0$  we have

$$A^{f_s}(u) = \int_{-\epsilon}^{\epsilon} (1 - \cos(ux)) f_s(x) dx + \int_{(-\epsilon, \epsilon)^c} (1 - \cos(ux)) f_s(x) dx.$$

Since  $(1 - \cos(ux)) \geq 0$  this yields  $A^{f_s}(u) \geq \int_{-\epsilon}^{\epsilon} (1 - \cos(ux)) f_s(x) dx$ . By inserting the assumption on  $f_s$ , we obtain for  $\epsilon$  small enough

$$\begin{aligned} A^{f_s}(u) &\geq \int_{-\epsilon}^{\epsilon} (1 - \cos(ux)) \frac{C}{|x|^{1+Y}} dx + \int_{-\epsilon}^{\epsilon} (1 - \cos(ux)) g(x) dx \\ &\geq C \int_{-\epsilon}^{\epsilon} \frac{1 - \cos(ux)}{|x|^{1+Y}} dx - \left| \int_{-\epsilon}^{\epsilon} (1 - \cos(ux)) g(x) dx \right|. \end{aligned}$$

For the first integral, a computation similar to (27) in the proof of Proposition 6.5 yields

$$\begin{aligned} \int_{-\epsilon}^{\epsilon} \frac{1 - \cos(ux)}{|x|^{1+Y}} dx &\geq c' \int_{|ux| \leq \pi} \frac{x^2 u^2}{|x|^{1+Y}} dx - C_1(\epsilon) \\ &= c' |u|^Y \int_{|x| \leq \pi} |x|^{1-Y} dx - C_1(\epsilon) = C_2 |u|^Y - C_1(\epsilon) \end{aligned}$$

with the positive constants  $C_1$  and  $C_2(\epsilon)$  given by  $C_1(\epsilon) = 2 \int_{(-\epsilon, \epsilon)^c} \frac{1}{|x|^{1+Y}} dx$  and  $C_2 := c' \int_{|x| \leq \pi} |x|^{1-Y} dx$ . In order to find an upper bound for the second integral, let us assume  $\epsilon < 1$ . Then  $|x|^{-1-Y+\delta} \leq |x|^{-1-Y+\delta'}$  with  $\delta' := \min\{Y/2, \delta\}$  for  $|x| < \epsilon$ . Arguing along the same lines as in the proof of equation (28) yields since  $\delta' < Y$

$$0 \leq \int_{-\epsilon}^{\epsilon} \frac{1 - \cos(ux)}{|x|^{1+Y-\delta}} dx \leq \int_{-\epsilon}^{\epsilon} \frac{1 - \cos(ux)}{|x|^{1+Y-\delta'}} dx \leq C(\epsilon) |u|^{Y-\delta'}$$

for some constant  $C(\epsilon) > 0$ . Fixing some appropriate  $\epsilon > 0$  we have

$$A^{f_s}(u) \geq C_1 |u|^Y - C_2 (1 + |u|^{Y-\delta'})$$

for a strictly positive constant  $C_1, C_2 \geq 0$  and  $0 < \delta' < Y$ .

Proof of c): The first equality of the assertion is given by equation (18). For every  $u \in \mathbb{R}$  we have  $|A^{f_{as}}(u)| \leq \int |ux - \sin(ux)| |f_{as}(x)| dx$  and if we choose  $\epsilon > 0$  small enough, Lemma 4.5 allows us to conclude

$$\begin{aligned} &\int_{(-\epsilon, \epsilon)^c} |ux - \sin(ux)| |f_{as}(x)| dx \\ &\leq |u| \int_{(-\epsilon, \epsilon)^c} |x| |f_{as}(x)| dx + \int_{(-\epsilon, \epsilon)^c} |f_{as}(x)| dx \\ &\leq |u| \int_{(-\epsilon, \epsilon)^c} |x| |f_s(x)| dx + \int_{(-\epsilon, \epsilon)^c} |f_s(x)| dx \\ &= |u| \int_{(-\epsilon, \epsilon)^c} |x| F(dx) + \int_{(-\epsilon, \epsilon)^c} F(dx) \\ &=: C_1(\epsilon) |u| + C_2(\epsilon) \end{aligned}$$

with nonnegative constants  $C_1(\epsilon)$  and  $C_2(\epsilon)$ . From the assumption on  $f_{as}$  we get

$$\begin{aligned}
& \int_{-\epsilon}^{\epsilon} |ux - \sin(ux)| |f_{as}(x)| \, dx \\
& \leq C(\epsilon) \int_{-\epsilon}^{\epsilon} |ux - \sin(ux)| \frac{1}{|x|^{1+Y}} \, dx \\
& = C(\epsilon) |u|^Y \int_{-\epsilon|u|}^{\epsilon|u|} \frac{|x - \sin(x)|}{|x|^{1+Y}} \, dx \\
& = 2C(\epsilon) |u|^Y \left( \int_0^1 \frac{x - \sin(x)}{|x|^{1+Y}} \, dx + \int_1^{\epsilon|u|} \frac{x - \sin(x)}{|x|^{1+Y}} \, dx \right),
\end{aligned}$$

where the first integral is finite since  $Y < 2$ . As before, the second integral is negative for  $\epsilon|u| < 1$  and for  $1 < \epsilon|u|$  we have

$$\int_1^{\epsilon|u|} \frac{x}{|x|^{1+Y}} \, dx = \int_1^{\epsilon|u|} x^{-Y} \, dx = \frac{\epsilon^{1-Y}}{1-Y} |u|^{1-Y} - \frac{1}{1-Y}$$

since  $Y \neq 1$  and

$$- \int_1^{\epsilon|u|} \frac{\sin(x)}{|x|^{1+Y}} \, dx \leq \int_1^{\epsilon|u|} x^{-1-Y} \, dx = -\frac{|u|^{-Y}}{Y\epsilon^Y} + \frac{1}{Y} \leq C_3(\epsilon)(1 + |u|^{-Y})$$

with some constant  $C_3(\epsilon) > 0$ . Combining these estimates and fixing some  $\epsilon > 0$ , we obtain the assertion of part c).

Proof of d): Since  $f_{as}$  is antisymmetric and  $\int |xf_{as}(x)| \, dx \leq \int |x|f(x) \, dx < \infty$  by Lemma 4.5, we obtain

$$A^{f_{as}}(u) = i\Im \left( A^f(u) \right) = i \int \sin(ux) f_{as}(x) \, dx - iu \int x f(x) \, dx.$$

Furthermore since the drift is given by  $\int x f(x) \, dx$  we have

$$\left| \Im(A(u)) \right| = \left| \int \sin(ux) f_{as}(x) \, dx \right| \leq \int_{(-\epsilon, \epsilon)^c} |f_{as}(x)| \, dx + \int_{-\epsilon}^{\epsilon} |\sin(ux)| |f_{as}(x)| \, dx$$

hence by the assumption on  $f_{as}$  we obtain

$$\begin{aligned}
\left| \Im(A(u)) \right| & \leq F((-\epsilon, \epsilon)^c) + C(\epsilon) \int_{-\epsilon}^{\epsilon} \frac{|\sin(ux)|}{|x|^{1+Y}} \, dx \\
& \leq C_1(\epsilon) + C(\epsilon) |u|^Y \int_{-\infty}^{\infty} \frac{|\sin(x)|}{|x|^{1+Y}} \, dx \\
& = C_1(\epsilon) + C_2(\epsilon) |u|^Y
\end{aligned}$$

with positive constants  $C(\epsilon)$ ,  $C_1(\epsilon)$  and  $C_2(\epsilon)$ . Choosing  $\epsilon > 0$  yields the result.  $\square$

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